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# THE HESTOCK AND HENSTOCK DELTA INTEGRALS

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Abstract. In this paper, we study the Henstock delta integral, which generalizes the Henstock integral. In particular, we study the relation between the Henstock and Henstock delta integrals.

## 1. Introduction and preliminaries

The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [2].

In this paper, we investigate the relation between the Henstock and Henstock delta integrals.

First, we introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

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For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

## 2. The Henstock and Henstock delta integrals

Definition 2.1. ([2]) $\delta = (\delta_L, \delta_R)$ is a $\triangle$-gauge on $[a, b]_{\mathbb{T}}$ by $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

Definition 2.2. ([2]) A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a Henstock partition of $[a, b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$.

For Henstock partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$, we write

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

Definition 2.3. ([2]). A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock delta integrable (or $H_\triangle$-integral) on $[a, b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a $\triangle$-gauge $\delta$ on $[a, b]_{\mathbb{T}}$ such that

$$\left| S(f, \mathcal{D}) - A \right| < \epsilon$$

for every $\delta$-fine Henstock partition $\mathcal{D}$ of $[a, b]_{\mathbb{T}}$. A number A is called the $H_\triangle$-integral of $f$ on $[a, b]_{\mathbb{T}}$, and we write $A = (H_\triangle) \int_a^b f \triangle t$.

Recall that $f : [a, b] \to \mathbb{R}$ is Henstock integrable (or H-integrable) on $[a, b]$ if there exists a number A such that for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ on $[a, b]$ such that

$$\left| S(f, \mathcal{P}) - A \right| < \epsilon$$

for every $\delta$-fine Henstock partition $\mathcal{P}$ of $[a, b]$.

Theorem 2.4. *A function $f : [a, b] \to \mathbb{R}$ is H-integrable on $[a, b]$ if and only if $f$ is $H_\triangle$-integrable on $[a, b]$.*

*Proof.* Let $f$ be H-integrable on $[a,b]$ and let $\epsilon > 0$. Then there exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\left|S(f,\mathcal{P}) - (H)\int_a^b f\right| < \epsilon$$

for every $\delta$-fine Henstock partition $\mathcal{P}$ of $[a,b]$.

Define $\delta^* = (\delta_L, \delta_R)$ by $\delta_L(t) = \delta_R(t) = \frac{\delta}{2}$ for each $t \in [a,b]$. Assume that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a $\delta^*$-fine partition of $[a,b]$. Then $\mathcal{P}$ is $\delta$-fine and

$$\left|S(f,\mathcal{P}) - (H)\int_a^b f\right| < \epsilon.$$

Hence $f$ is $H_\triangle$-integrable on $[a,b]$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f$. Conversely, assume that $f$ is $H_\triangle$-integrable on $[a,b]$ and let $\epsilon > 0$. Then there exists a $\triangle$-gauge $\delta^* = (\delta_L, \delta_R)$ such that

$$\left|S(f,\mathcal{P}) - (H_\triangle)\int_a^b f\triangle t\right| < \epsilon$$

for every $\delta^*$-fine partition $\mathcal{P}$ of $[a,b]$.

Define

$$\delta(t) = \begin{cases} \min\{\delta_L(t), \delta_R(t)\} & \text{if} \quad \mathrm{t} \in (a,b) \\ \delta_R(t) \quad \text{if} \quad t = a \\ \delta_L(t) \quad \text{if} \quad t = b. \end{cases}$$

Assume that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a $\delta$-fine partition of $[a,b]$.

Then $\mathcal{P}$ is $\delta^*$-fine and

$$\left|S(f,\mathcal{P}) - (H_\triangle)\int_a^b f\triangle t\right| < \epsilon.$$

Hence $f$ is H-integrable on $[a,b]$ and $(H)\int_a^b f = (H_\triangle)\int_a^b f\triangle t$. □

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a,b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for some} \quad k \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}}. \end{cases}$$

Then we have the following theorem.

THEOREM 2.5. *If $f^* : [a,b] \to \mathbb{R}$ is H-integrable on $[a,b]$, then $f : [a,b]_T \to \mathbb{R}$ is $H_\triangle$-integrable on $[a,b]_T$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f^*$.*

*Proof.* Let $f^* : [a,b] \to \mathbb{R}$ be H-integrable on $[a,b]$ and let $\epsilon > 0$. By theorem 2.4, there exists a $\triangle$-gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]$ such that

$$\left|S(f^*, \mathcal{P}) - (H)\int_a^b f^*\right| < \frac{\epsilon}{2}$$

for every $\delta$-fine Henstock partition $\mathcal{P}$ of $[a,b]$.

Define a $\triangle$-gauge $\delta^* = (\delta_L^*, \delta_R^*)$ on $[a,b]_{\mathbb{T}}$ by

$$\delta_L^*(t) = \delta_L(t)$$

$$\delta_R^*(t) = \begin{cases} \delta_R(t) \text{ if } t \text{ is a right-dense point of } [a,b]_{\mathbb{T}} \\ \sigma(t) - t \text{ if } t \text{ is a right-scattered point of } [a,b]_{\mathbb{T}}. \end{cases}$$

Let $\mathcal{D} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ be a $\delta^*$-fine partition of $[a,b]_{\mathbb{T}}$. Define $A = \{i : \xi_i$ is a right $-$ scattered point of $[a,b]_T$ and

$[\xi_i, \sigma(\xi_i)] \subset [t_{i-1}, t_i]\}$,

$A_1 = \{i \in A : t_{i-1} = \xi_i\}, A_2 = \{i \in A : t_{i-1} < \xi_i\}$, and

$B = \{1, 2, 3, \cdots, n\} - A$.

Let $D_0 = \{(\xi_i, [t_{i-1}, t_i]) : i \in B\}$. Then $D_0$ is a $\delta$-fine partial partition of $[a,b]$. For each $i \in A$, there is a $\delta$-fine partition $D_i'$ of $[\xi_i, \sigma(\xi_i)]$ such that

$$\left|S(f^*, D_i') - (H)\int_{\xi_i}^{\sigma(\xi_i)} f^*\right| < \frac{\epsilon}{2n}.$$

For each $i \in A$, let $D_i = \begin{cases} D_i' \text{ if } i \in A \\ D_i' \bigcup \{(\xi_i, [t_{i-1}, \xi_i])\} \text{ if } i \in A_2. \end{cases}$

Then $\mathcal{P} = D_0 \cup [\bigcup_{i \in A} D_i]$ is a $\delta$-fine partition of $[a,b]$ and we have

$$\begin{aligned} &\left|S(f, D) - (H)\int_a^b f^*\right| \\ &\leq \left|S(f, D) - S(f^*, \mathcal{P})\right| + \left|S(f^*, \mathcal{P}) - (H)\int_a^b f^*\right| \\ &\leq \sum_{i \in A_1} \left|f(\xi_i)(t_i - t_{i-1}) - S(f^*, D_i)\right| \\ &\qquad + \sum_{i \in A_2} \left|f(\xi_i)(t_i - t_{i-1}) - S(f^*, D_i)\right| + \frac{\epsilon}{2} \\ &= \sum_{i \in A_1} \left|f(\xi_i)(\sigma(\xi_i) - \xi_i) - S(f^*, D_i')\right| \\ &\qquad + \sum_{i \in A_2} \left|f(\xi_i)(\sigma(\xi_i) - \xi_i) - S(f^*, D_i')\right| + \frac{\epsilon}{2} \end{aligned}$$

$$= \sum_{i\in A} \left|(H)\int_{\xi_i}^{\sigma(\xi_i)} f^* - S(f^*, D_i')\right| + \frac{\epsilon}{2}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Hence, $f$ is $H_\triangle$-integral on $[a,b]_{\mathbb{T}}$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f^*$. $\square$

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